

# Exponential localization of singular vectors in spatiotemporal chaos

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In a dynamical system the singular vector (SV) indicates which perturbation will exhibit maximal growth after a time interval  $\tau$ . We show that in systems with spatiotemporal chaos the SV exponentially localizes in space. Under a suitable transformation, the SV can be described in terms of the Kardar-Parisi-Zhang equation with periodic noise. A scaling argument allows us to deduce a universal power law  $\tau^{-\gamma}$  for the localization of the SV. Moreover the same exponent  $\gamma$  characterizes the finite- $\tau$  deviation of the Lyapunov exponent in excellent agreement with simulations. Our results may help improving existing forecasting techniques.

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## I. INTRODUCTION

Lyapunov analysis is one of the main tools to quantitatively characterize dynamical systems [1]. In particular, the largest Lyapunov exponent (LE)  $\lambda$  is a fundamental quantity that allows one to address, at some basic level, the question of the predictability of a system. A positive  $\lambda$  implies chaotic behavior: neighboring initial conditions diverge in time at an exponential rate  $\sim \exp(\lambda t)$ . Actually, the LE is an asymptotic magnitude in the sense that it is defined in the infinite-time limit.

However, it has been since long recognized that the infinite-time limit is often irrelevant to address many important questions. Therefore, the *finite-time* LE (to be defined below) was introduced as an essential tool to deal with problems such as predictability [2] and shadowability [3] of dynamical systems. Correspondingly, the singular vector (SV) (also referred to as the finite-time Lyapunov vector) is defined as the perturbation that will show the maximal expansion after some *finite* time  $\tau$ .

Singular vectors are typically used in operative models for weather prediction [4, 5]. Every day the ECMWF (European Centre for Medium-Range Weather Forecasts) [6] computes a set of singular vectors to perform an ensemble forecasting of the European weather [4]. Also, in a different context, SVs have been found to be useful to study advection of passive particles in fluids [7].

It is a well-known result [8] that as the time horizon  $\tau$  increases the SV exponentially approaches the so-called forward Lyapunov vector (FLV) [9]. For low dimensional systems our understanding in these simple terms might be satisfactory. However, in extended systems it is clear that spatiotemporal correlations should have an important effect in the way this limit is achieved. How this convergence of the SV toward the FLV occurs and what generic system-independent features (if any) exist are important open questions in the case of chaotic extended systems. Indeed, this is also of practical and theoretical importance in the field of atmospheric sciences [10, 11], because it is directly related with the problem of realistic forecasting within a given finite time horizon  $\tau$ .

In this paper we analyze the structure of the SV in extended systems with spatiotemporal chaos. We show that the SV localizes exponentially in space around one center (determined by the present state of the system and by the horizon  $\tau$ ). We find that the localization strength generically scales with the

observation time as a power-law  $\tau^{-\gamma}$ . Moreover, we propose a stochastic field-theoretical description of the evolution of the SV, which allows us to give an analytical prediction for the universal exponent  $\gamma$  and the finite- $\tau$  deviation of the LE, in good agreement with simulations in systems of different natures, including coupled-map lattices and differential equations. Numerical results together with general scaling arguments suggest our results are generic for a wide class of systems exhibiting space-time chaos.

## II. BASIC DEFINITIONS

For any given dynamical system, infinitesimal perturbations  $\delta \mathbf{u}(t)$  are governed by linear equations (tangent space) such that the perturbation after some  $\tau$  can be expressed via a linear operator (resolvent or propagator):  $\delta \mathbf{u}(t + \tau) = \mathbf{M}(t + \tau, t) \delta \mathbf{u}(t)$ .

We are interested here in the SV, which is defined as the perturbation in tangent space at time  $t$  that gets amplified the most at some future time  $t + \tau$ . In a more mathematical form, let  $\mathbf{s}_\tau(t)$  denotes the SV for a time horizon  $\tau$ , then we have

$$\mathbf{s}_\tau(t) = \arg \max_{\delta \mathbf{u}(t)} \frac{\langle \mathbf{M}(t + \tau, t) \delta \mathbf{u}(t) \cdot \mathbf{M}(t + \tau, t) \delta \mathbf{u}(t) \rangle}{\langle \delta \mathbf{u}(t) \cdot \delta \mathbf{u}(t) \rangle} \quad (1)$$

where  $\langle \mathbf{x} \cdot \mathbf{y} \rangle$  denotes the scalar product. The “optimal” perturbation  $\mathbf{s}_\tau(t)$  is the SV and it depends on both the time  $t$  (that determines the present state of the system), and the optimization time  $\tau$ . Using the adjoint operator  $\mathbf{M}^*(t + \tau, t)$  defined by  $\langle \mathbf{M}(t + \tau, t) \delta \mathbf{u}(t) \cdot \mathbf{M}(t + \tau, t) \delta \mathbf{u}(t) \rangle = \langle \delta \mathbf{u}(t) \cdot \mathbf{M}^*(t + \tau, t) \mathbf{M}(t + \tau, t) \delta \mathbf{u}(t) \rangle$ , the problem posed in Eq. (1) can be solved by finding the eigenvector of  $\mathbf{M}^*(t + \tau, t) \mathbf{M}(t + \tau, t)$  with the largest eigenvalue (all of them are real and positive). For the Euclidean scalar product used throughout this paper we have  $\mathbf{M}^* = \mathbf{M}^T$ .

The iterative application of the operator  $\Phi(t + \tau, t) \equiv \mathbf{M}^*(t + \tau, t) \mathbf{M}(t + \tau, t)$  to an arbitrary vector (the power method), generically converges to the wanted eigenvector  $\mathbf{s}_\tau(t)$ , which satisfies the eigenvalue problem  $\Phi(t + \tau, t) \mathbf{s}_\tau(t) = \mu_\tau(t) \mathbf{s}_\tau(t)$ . Then, the finite-time LE is typically defined as  $\lambda_\tau(t) = (2\tau)^{-1} \ln \mu_\tau(t)$  [8, 12, 13], which depends on  $t$  and on the optimization time  $\tau$ . According to Oseledec’s theorem [14] in the limit  $\tau \rightarrow \infty$  with probability

one  $\lambda_\tau(t)$  converges to a time-independent quantity—namely, the largest LE of the chaotic attractor  $\lim_{\tau \rightarrow \infty} \lambda_\tau(t) = \lambda$ . An important observation is in order:  $\lambda$  is usually obtained from the evolution of a perturbation grown from the past, instead of considering how a perturbation will evolve in the future. Thus a perturbation evolved since the remote past is the “most unstable” eigenvector of  $\mathbf{M}(t, -\infty)\mathbf{M}^*(t, -\infty)$  [15], which has been called backward Lyapunov vector (BLV) [9]. Conversely, the FLV is eigenvector of  $\mathbf{M}^*(\infty, t)\mathbf{M}(\infty, t)$ .

In order to gain some numerical insight into the problem we have studied the behavior of the SV in two prototypical dynamical systems exhibiting spatiotemporal chaos. The first example we consider is a CML in a ring, which is discrete in space ( $x = 1, \dots, L$ ) and time ( $t = 0, 1, \dots, \infty$ ):

$$u_x(t+1) = \epsilon f(u_{x+1}(t)) + \epsilon f(u_{x-1}(t)) + (1-2\epsilon)f(u_x(t)), \quad (2)$$

with the logistic map  $f(\varrho) = 4\varrho(1-\varrho)$ , and coupling strength  $\epsilon = 1/3$ .

The second model we consider here is an example of a chaotic system described by differential equations. We study the model proposed by Lorenz in 1996 (L96) [16] as a toy model in the context of weather dynamics. We consider the variables  $y_x$  defined in a ring geometry, and the evolution equations

$$\frac{d}{dt}y_x = -y_x - y_{x-1}(y_{x-2} - y_{x+1}) + F. \quad (3)$$

with  $F = 8$ . We used the Euler method for the integration with  $\Delta t = 10^{-4}$ .

### III. SINGULAR VECTOR SURFACE: LOCALIZATION, PATTERNS AND CORRELATIONS

An important feature of singular and Lyapunov vectors is that their amplitudes are much localized in space. Localization of SVs was observed long time ago in simple meteorological models [12, 17], however, to our knowledge, a description of the localization profile in space or the dynamics does not exist even at a qualitative level. In contrast, the localization properties of the BLV are much better understood. Pikovsky *et al.* [18, 19] demonstrated that, in a large class of one-dimensional systems with spatiotemporal chaos, the BLV can be mapped into a roughening surface that belongs to the universality class of the Kardar-Parisi-Zhang (KPZ) equation [20]. The solutions of the KPZ equation present universal scaling exponents in space and time [20]. In particular, and as a result, one can conclude that the BLV in one dimension is sub-exponentially localized, with its magnitude decaying in space as  $\sim \exp(-k\sqrt{x})$  from the localization site. These results apply to the FLV as well because the operator  $\mathbf{M}^*$  has the same coarse-grained long-scale structure than  $\mathbf{M}$ , and thus the FLV surface belongs to the KPZ universality class too.

The spatial structure of a Lyapunov vector is better resolved by making a logarithmic transformation [18, 19, 21, 22]. In the same spirit, expressing the vector components as  $s_\tau(t) =$

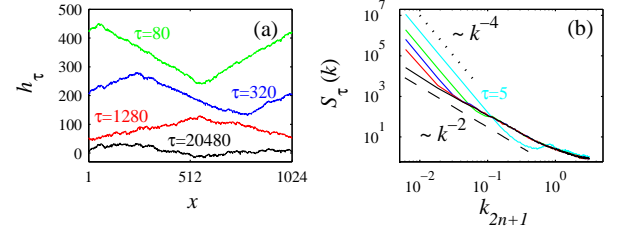


FIG. 1: (Color online) (a) SV surfaces of the CML model for different values of  $\tau$  ( $L = 1024$ ). Curves have been shifted to improve their visibility. (b) Structure factor for different values of  $\tau$  (averaged over 500 realizations).

$[s_\tau(x, t)]_{x=1}^L$  we define a SV surface via the Hopf-Cole transformation:  $h_\tau(x, t) = \ln |s_\tau(x, t)|$ . In this representation the finite- $\tau$  LE becomes the average velocity of the surface:  $\lambda_\tau(t) = (\tau L)^{-1} \sum_{x=1}^L \ln |\mathbf{M}(t+\tau, t)s_\tau(x, t)|/s_\tau(x, t)|$ .

Figure 1(a) shows typical SV surfaces for different values of  $\tau$  in the case of the CML model (2). We obtained similar behavior for the L96 model (3). We observe that for  $\tau$  larger than some threshold  $\tau_\times(L)$  the corresponding surface has approximately a random-walk profile; or in terms of the SV itself we have stretched-exponential localization for a time horizon  $\tau \gg \tau_\times(L)$ , as anticipated above. Remarkably, for  $\tau < \tau_\times(L)$  we find that the height profile of the SV surface is faceted with superimposed fluctuations at small scale. Note that a triangular SV surface corresponds to an exponential localization of the SV.

The triangular pattern progressively dies out for longer times  $\tau$ . This means that the asymptotic convergence to the FLV is reached for practical purposes in a finite time: when we look for the perturbation at present time  $t$  that will exhibit the largest amplification at a future time  $t + \tau$ , there is a characteristic time  $\tau_\times(L)$  that separates the genuine finite-time regime from the quasi-infinite-time regime. In the latter regime ( $\tau > \tau_\times$ ) the SV has almost collapsed into the FLV.

Quantitative information about the spatial correlations of the SV can be obtained calculating the structure factor:  $S_\tau(k) = \lim_{t \rightarrow \infty} \langle \hat{h}_\tau(k, t) \hat{h}_\tau(-k, t) \rangle$ , where  $\hat{h}_\tau(k, t) = L^{-1/2} \sum_x h_\tau(x, t) \exp(ikx)$ , and the bracket indicates average over realizations. Figure 1(b) shows the results for different values of the temporal horizon  $\tau$ . For large  $\tau$ , one recovers the typical spatial correlations corresponding to the FLV,  $S_\tau(k) \sim k^{-2}$ , as expected from the previous discussion. Contrastingly, for  $\tau < \tau_\times(L)$  we find  $S_\tau(k) \sim k^{-4}$  at small wave numbers (due to the symmetry of the triangular profile only wavenumbers  $k_n = 2\pi n/L$  with  $n$  odd contribute significantly to the structure factor). The coarse-grained slope of the triangle  $a$ , and the surface width  $W_\tau^2(L) = L^{-1} \sum_x h_\tau(x, t)^2 - [L^{-1} \sum_x h_\tau(x, t)]^2$  serve to quantify the localization strength. The following relation holds for small  $k$  and  $\tau < \tau_\times(L)$ :

$$S_\tau(k_n \rightarrow 0) \cong \frac{16\langle a^2 \rangle}{Lk_n^4} \cong \frac{768\langle W_\tau^2(L) \rangle}{L^3k_n^4}$$

for  $n$  odd.

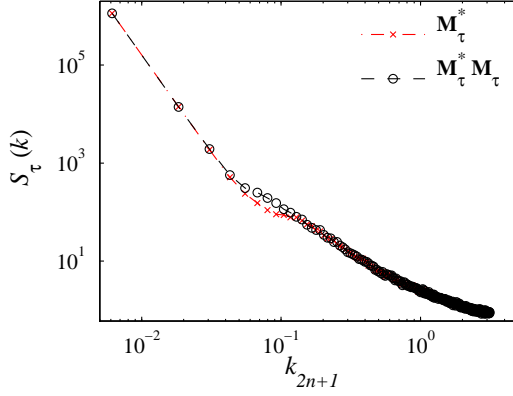


FIG. 2: (Color online) Structure factor of the eigenvectors of  $\mathbf{M}_\tau^*$  and  $\mathbf{M}_\tau^* \mathbf{M}_\tau$  for  $\tau = 160$  in the CML model after average over 500 realizations. The long scale structure (small  $k$ ) of the eigenvector of  $\mathbf{M}_\tau^* \mathbf{M}_\tau$  is well captured by the eigenvector of  $\mathbf{M}_\tau^*$  alone.

#### IV. STOCHASTIC FIELD THEORY FOR SINGULAR VECTORS

SVs are the asymptotic vectors computed by repetitive application of the operator  $\Phi(t+\tau, t) = \mathbf{M}^*(t+\tau, t)\mathbf{M}(t+\tau, t)$ , to an arbitrary initial vector. This makes the SV similar, albeit not exactly equivalent, to the BLV of a periodic orbit. In fact, for temporally periodic but spatially chaotic states the evolution in tangent space is mapped exactly onto an one-dimensional localization problem and it can be rigorously proved that all the eigenvectors are exponentially localized [23]. For the FLV to be computed, and because of the  $\tau \rightarrow \infty$  limit, only the  $\mathbf{M}^*$  operator needs to be used [15]. However, the structure of the operator  $\Phi$  is obviously different from that of  $\mathbf{M}^*$ . In Fig. 2 we compare the structure factors obtained with the example of the CML when computing the surfaces associated to the eigenvectors of  $\Phi(t+\tau, t)$  (the actual singular vector) and  $\mathbf{M}^*(t+\tau, t)$  (which can be viewed as the BLV of an unstable periodic orbit of period  $\tau$ ). One may see that, in statistical terms, the large length scale behavior of the main eigenvector of  $\Phi(t+\tau, t)$  is well reproduced using operator  $\mathbf{M}^*(t+\tau, t)$  alone. This fact justifies considering a minimal model (for the long scale structure of the SV) consisting of a stochastic equation with *periodic* noise.

We now write a minimal Langevin model to understand in the simplest terms the structure of the SV surface on long wavelengths. We propose a modification of the KPZ equation considering a periodic noise (PNKPZ):

$$\partial_t h_\tau(x, t) = \zeta_\tau(x, t) + [\partial_x h_\tau(x, t)]^2 + \partial_{xx} h_\tau(x, t), \quad (4)$$

where one simply assumes  $\zeta_\tau$  to be a random noise with period  $\tau$  [i.e.  $\zeta_\tau(x, t) = \zeta_\tau(x, t + \tau)$ ] and a  $\delta$ -correlator,  $\langle \zeta_\tau(x, t) \zeta_\tau(x', t') \rangle = 2\sigma \delta(x - x') \delta(t - t')$ , for  $|t - t'| < \tau$ .

Figure 3 shows the results of our numerical integration of Eq. (4); space and time were discretized ( $\Delta x = 100\Delta t = 1$ ) and a noise intensity  $\sigma = 0.5$  was used. Other intensities can be used and this, of course, has no effect on the results. One can see that the solutions of (4) adopt a triangular pattern, akin

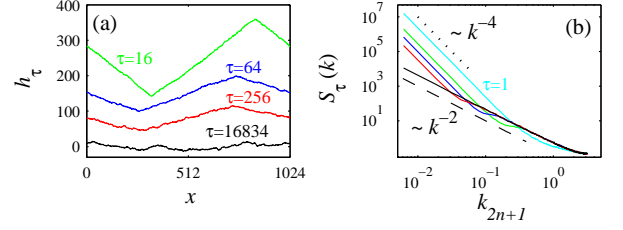


FIG. 3: (Color online) (a) Asymptotic solutions of the PNKPZ equation for different values of  $\tau$  ( $L = 1024$ ). Curves are arbitrarily shifted to improve their visibility. (b) Structure factors for different values of  $\tau$  (averages were done over 500 realizations).

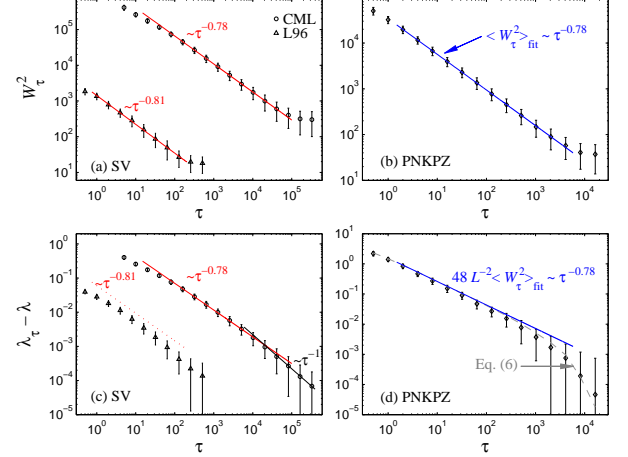


FIG. 4: (Color online) (a,b) Surface width vs.  $\tau$ . (a) SVs of CML and L96 models for  $L = 4096$  and  $L = 512$ , respectively. (b) PNKPZ for  $L = 1024$ . (c,d) Finite- $\tau$  deviation of the LE. Data of the L96 model have been shifted two decades downwards to avoid overlap of data sets. In all panels symbols mark mean values, and bars indicate the statistical dispersion (100 realizations were done for the L96 model and 500 for the other models).

to those shown in Fig. 1 for the SV surface of the deterministic model. The triangular structure flattens as  $\tau$  increases and eventually disappears above some threshold value  $\tau_\times(L)$ .

In Fig. 4(a,b) we plot the surface width  $W_\tau^2$  as a function of  $\tau$ . Figure 4(a) was generated from the SV surfaces of the chaotic systems in Eqs. (2)-(3), while Fig. 4(b) was obtained from the numerical integration of the stochastic PNKPZ equation (4). For not too large values of  $\tau$  we find a region of scaling where the width is dominated by the triangular structure. We find

$$\langle W_\tau^2 \rangle \sim \tau^{-\gamma}$$

The values of  $\gamma$  with a 95% of confidence are:  $\gamma = 0.78 \pm 0.01$  for the CML,  $\gamma = 0.81 \pm 0.03$  for the L96 model, and  $\gamma = 0.78 \pm 0.01$  for the PNKPZ equation. This strongly suggests that  $\gamma$  is a universal exponent.

In all cases we expect the scaling to break down for large  $\tau$ , as can be inferred from the fact that the stochastic model (4) converges to the KPZ problem as  $\tau \rightarrow \infty$ . This translates into the convergence of the SV to the FLV as the time horizon

$\tau$  tends to infinity. The existence of the characteristic time  $\tau_{\times}(L)$  means that the FLV is virtually reached in a finite time for a finite-size system.

For the theoretical analysis to follow, it is worth to remark that triangular solutions also appear in the case of the quenched columnar KPZ (QCKPZ) equation discussed in Ref. [24], that corresponds to Eq. (4) but with the periodic noise  $\zeta(x, t)$  replaced with a quenched columnar disorder  $\tilde{\zeta}(x)$ . Note that in the limit  $\tau \rightarrow 0$  of the PNKPZ equation [cf. (4)] one recovers the QCKPZ equation. However, for a finite  $\tau$  correlations spread and one may conjecture that the problem can be seen effectively as a QCKPZ equation where the disorder  $\tilde{\zeta}(x)$  has a finite correlation length. The asymptotic solutions of the QCKPZ equation have been described as a triangular pattern (with superimposed random fluctuations) and exhibit scale-invariant roughness  $\langle W_{\tau=0}^2(L) \rangle \sim L^{2\alpha_{QC}}$  [24].

The existence of these two limiting dynamics —namely, KPZ for  $\tau \rightarrow \infty$  and QCKPZ for  $\tau \rightarrow 0$ — describes the behavior of the PNKPZ equation and can be exploited to derive an analytical expression for the exponent  $\gamma$  by resorting to scaling arguments as follows. On the one hand, the width of the PNKPZ surface should scale as  $\langle W_{\tau}^2(L) \rangle \sim L^{2\alpha_{KPZ}}$  for  $\tau \gg \tau_{\times}(L)$ , and it should make a crossover to  $\langle W_{\tau}^2(L) \rangle \sim L^{2\alpha_{QC}}\tau^{-\gamma}$  for  $\tau \ll \tau_{\times}(L)$ . Therefore, these two limiting behaviors are separated by the crossover time  $\tau_{\times}(L) \sim L^{2(\alpha_{QC}-\alpha_{KPZ})/\gamma}$ . On the other hand, the characteristic time  $\tau_{\times}(L)$  should also correspond to the typical time that an initially flat KPZ surface needs to reach the statistically stationary regime in a system of size  $L$ , so we have  $\tau_{\times}(L) \sim L^{z_{KPZ}}$  [20]. Equating both expressions we arrive at

$$\gamma = \frac{2(\alpha_{QC} - \alpha_{KPZ})}{z_{KPZ}}, \quad (5)$$

which is in principle valid in *any* dimension and can be compared with our numerical simulations. Inserting the critical exponents in one dimension — $\alpha_{KPZ} = 1/2$  and  $z_{KPZ} = 3/2$  [20]; and  $\alpha_{QC} = 1.07 \pm 0.05$  [24]— we get  $\gamma = 0.76 \pm 0.07$  in good agreement with our simulations (our direct estimation of  $\alpha_{QC}$  for the PNKPZ equation (4) with  $\tau = 16$  yields  $\alpha_{QC} = 1.08 \pm 0.02$ , so  $\gamma = 0.78 \pm 0.03$ ).

## V. FINITE-TIME LYAPUNOV EXPONENT

A very interesting outcome of our analysis is the existence of a scaling law for the finite-time LE. The average velocity of the PNKPZ surface satisfies  $\lambda_{\tau}(t) = \langle L^{-1} \sum_x (\partial_x h_{\tau}(x, t))^2 \rangle_{\tau}$ , where in good approximation the noise term has been assumed to average out within a period. Existing results on QCKPZ [24] indicate that there is a complete separation in two independent components such that  $h$  can be constructed as the sum of a triangular pattern with a tilt  $\pm ax$  and the solutions in  $\tau \rightarrow \infty$ . After some algebra, one

obtains

$$\langle \lambda_{\tau} \rangle - \lambda = \langle a^2 \rangle = 48L^{-2}(\langle W_{\tau}^2 \rangle - \langle W_{\tau=\infty}^2 \rangle) \quad (6)$$

In Fig. 4(d) we test the validity of this expression by plotting  $\lambda_{\tau} - \lambda$  as a function of  $\tau$ . The dashed line arises inserting in (6) the mean values of  $W_{\tau}^2$  depicted in Fig. 4(b). The straight line is computed by inserting in the dominant contribution  $48L^{-2}\langle W_{\tau}^2 \rangle$  the fitting power law in Fig. 4(b):  $\langle W_{\tau}^2 \rangle_{\text{fit}} = 3.4 \times 10^4 \tau^{-0.78}$ . The agreement is, in our opinion, very good taking into account the simplicity of our arguments.

For a generic system with spatiotemporal chaos we may not expect to be able to obtain a formula relating the width of the SV surface and the finite- $\tau$  deviation of the LE, but still we may ask whether the  $\tau^{-\gamma}$  dependence is observed. Figure 4(c) indicates that this scaling for the LE exists in the same region where  $\langle W_{\tau}^2 \rangle \sim \tau^{-\gamma}$ , although the result is not conclusive for the L96 model because computer limitations did not allow us to study larger systems. Finally, we point out that for  $\tau > \tau_{\times}(L)$  the convergence of  $\lambda_{\tau}$  to  $\lambda$  becomes faster, precisely as  $\sim \tau^{-1}$  in accordance with the standard asymptotic behavior for low-dimensional systems [8]. This means that the spatial degrees of freedom slow down the convergence to the LE in extended dynamical systems as compared with the low-dimensional case.

## VI. CONCLUSIONS

We have shown that for a large family of systems the SV exhibits exponential localization. We claim that this spatial structure can be described in terms of the solutions of the a stochastic equation, related to KPZ equation, in which the noise is periodic or quenched. In terms of the associated surfaces, the SV converges toward the FLV following a power law with a universal exponent  $-\gamma$  given by Eq. (5). In particular, for one-dimensional systems one has  $\gamma \simeq 0.78$ . Moreover, the same exponent characterizes the finite-time deviation of the LE. Even for spatio-temporal chaotic systems whose BLV (and FLV) does not belong to the universality class of KPZ, as occurs with Hamiltonian lattices [25], one may expect a qualitative similarity with the scenario presented in this work, but possibly with different exponents.

In forecasting applications  $\tau$  is the control parameter to calibrate the ensemble of SVs. Our results show how  $\tau$  quantifies (through the universal exponent  $\gamma$ ) the localization strength, and the exponential growth rate.

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